



# On a factorization of homeomorphisms of the circle possessing periodic points

Marek Cezary Zdun

*Institute of Mathematics, Pedagogical University, ul. Podchorążych 2, PL-30-084 Kraków, Poland*

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## Abstract

We prove that for every orientation-preserving homeomorphism  $F : S^1 \rightarrow S^1$  possessing periodic points of order  $n$  there exist a homeomorphism  $T : S^1 \rightarrow S^1$  such that  $T^n = \text{id}$  and a homeomorphism  $G : S^1 \rightarrow S^1$  without periodic points except fixed points such that

$$F = T^q \circ G$$

for an integer  $1 \leq q < n$  relatively prime with  $n$ . We apply this result to determine all continuous iterative roots of homeomorphisms of the circle which have periodic points.

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**1.** In the present paper we shall investigate relations between homeomorphisms of the circle possessing periodic points and homeomorphisms of a compact interval. We begin by recalling the basic definitions and introducing some notations. Let us denote by  $S^1$  the unit circle on the complex plane. Define on  $S^1$  the following triple ordering relation. For any  $u, w, z \in S^1$  there exist unique  $t_1, t_2 \in [0, 1)$  such that  $we^{2\pi i t_1} = z$ ,  $we^{2\pi i t_2} = u$ . Define

$$u < w < z \quad \text{if and only if} \quad 0 < t_1 < t_2$$

(see [1,2]).

For any distinct elements  $u, z \in S^1$  the sets  $\overrightarrow{(u, z)} := \{w \in S^1 : u < w < z\}$ ,  $\overleftarrow{(u, z)} := \overrightarrow{(u, z)} \cup \{u\}$  and  $\overleftarrow{[u, z]} := \overrightarrow{(u, z)} \cup \{z\}$  are said to be *the arcs*.

Let  $A \subset S^1$  be a nonempty set. We say that a function  $F : A \rightarrow S^1$  preserves orientation if for any  $u, w, z \in A$  such that  $u < w < z$  we have  $F(u) < F(w) < F(z)$ .

For every orientation-preserving homeomorphism  $F : S^1 \rightarrow S^1$  there exists an increasing homeomorphism  $f : \mathbb{R} \rightarrow \mathbb{R}$ , which is unique up to a translation by an integer, such that

$$F(e^{2\pi i x}) = e^{2\pi i f(x)}, \quad x \in \mathbb{R},$$

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*E-mail address:* [mczdun@ap.krakow.pl](mailto:mczdun@ap.krakow.pl).

and

$$f(x+1) = f(x) + 1, \quad x \in \mathbb{R}.$$

The function  $f$  is called a *lift* of  $F$  (see e.g. [4]).

For every orientation-preserving homeomorphism  $F$  the number  $\alpha(F) \in [0, 1)$  defined by

$$\alpha(F) = \lim_{n \rightarrow \infty} \frac{f^n(x)}{n} \pmod{1}, \quad x \in \mathbb{R},$$

is called the *rotation number* of  $F$ . This number always exists and does not depend on  $x$  and the choice of the lift  $f$ .

It is well known (see for instance [4]) that a homeomorphism  $F: S^1 \rightarrow S^1$  has a periodic point if and only if its rotation number  $\alpha(F)$  is rational and positive. Then all periodic points have the same period (see [6,7]). More precisely, if  $\alpha(F) = \frac{q}{n}$ , where  $q, n \in \mathbb{N}$ ,  $n \geq 2$ , and  $\gcd(q, n) = 1$ , then the set  $\text{Per } F$  of all periodic points of  $F$  equals  $\text{Per}(F, n)$ , where

$$\text{Per}(F, n) := \{z \in S^1: F^n(z) = z, F^k(z) \neq z, 1 \leq k < n\}.$$

Moreover, there exists a unique  $p \in \{1, \dots, n-1\}$  such that  $pq \pmod{n} = 1$ . This number is called the *characteristic integer* of  $F$ . We shall write  $p = \text{char } F$ .

An orientation-preserving homeomorphism  $T: S^1 \rightarrow S^1$  such that

$$T^n(z) = z, \quad z \in S^1,$$

for  $n \geq 2$  is said to be a *Babbage function of order  $n$* . Every Babbage function is conjugated to a rotation (see [3,5]). The general construction of these homeomorphisms is given in [5].

**2.** In this section we will present the main result on factorization of homeomorphisms possessing periodic points. We begin by recalling the following results (see [7]).

**Lemma 1.** If  $a \in \text{Per}(F, n)$ ,  $\{a, F(a), \dots, F^{n-1}(a)\} = \{z_0, z_1, \dots, z_{n-1}\}$ , where  $z_0 = a$  and

$$\text{Arg} \frac{z_i}{z_0} < \text{Arg} \frac{z_{i+1}}{z_0} < 2\pi, \quad i = 0, \dots, n-2,$$

and  $F(a) = z_q$ , then  $\alpha(F) = \frac{q}{n}$ .

We shall use the following result (see [8]).

**Lemma 2.** If  $\text{Per}(F, n) \neq \emptyset$  for  $n \geq 2$ , then for every  $a \in \text{Per } F$ ,

$$\text{Arg} \frac{F^{kp}(a)}{a} < \text{Arg} \frac{F^{(k+1)p}(a)}{a}, \quad k = 0, \dots, n-2,$$

where  $p = \text{char } F$ .

Put

$$e_k := \exp\left(\frac{2\pi i k}{n}\right), \quad L_k := \overrightarrow{[e_k, e_{k+1}]}, \quad k = 0, \dots, n-1,$$

and

$$I_k := \overrightarrow{[F^{pk}(a), F^{p(k+1)}(a)]}, \quad k = 0, \dots, n-1,$$

where  $a \in \text{Per } F$  and  $p = \text{char } F$ .

**Lemma 3.** Let  $\alpha(F) = \frac{q}{n}$ ,  $\gcd(q, n) = 1$  and  $a \in \text{Per } F$ . Suppose that  $\Phi_k: I_k \rightarrow L_k$ ,  $k = 0, \dots, n-1$ , are arbitrary orientation-preserving homeomorphisms. Then

$$\Phi(z) := \Phi_k(z) \quad \text{for } z \in I_k,$$

is an orientation-preserving homeomorphism such that

$$\Phi(F^{pk}(a)) = e_k, \quad k = 0, \dots, n-1, \quad (1)$$

and

$$U(e_k) = e_{(k+q) \pmod n}, \quad k = 0, \dots, n-1, \quad (2)$$

where

$$U := \Phi \circ F \circ \Phi^{-1}. \quad (3)$$

**Proof.** The first assertion is an obvious consequence of Lemma 2. We have by (1),

$$\begin{aligned} U(e_k) &= \Phi \circ F \circ \Phi^{-1}(e_k) = \Phi(F(F^{pk}(a))) = \Phi(F^{pk+1}(a)) \\ &= \Phi(F^{pk+pq}(a)) = \Phi(F^{p(k+q)}(a)) = e_{(k+q) \pmod n}, \end{aligned}$$

since  $pq = 1 \pmod n$ .  $\square$

Put

$$R(z) := e_1 z, \quad z \in S^1.$$

**Lemma 4.** Let a homeomorphism  $U : S^1 \rightarrow S^1$  fulfil (2) for  $1 \leq q < n$  with  $\gcd(q, n) = 1$  and

$$H(z) := \begin{cases} R^q \circ U^n(z), & z \in L_0, \\ R^q(z), & z \in S^1 \setminus L_0. \end{cases} \quad (4)$$

Then there exists a homeomorphism  $\Psi : S^1 \rightarrow S^1$  such that

$$\Psi(H(z)) = U(\Psi(z)), \quad z \in S^1, \quad (5)$$

and

$$\Psi(z) = z \quad \text{for } z \in L_0. \quad (6)$$

**Proof.** First note that  $H$  is a homeomorphism, since by (2),

$$R^q \circ U^n(e_0) = R^q(e_0) \quad \text{and} \quad R^q \circ U^n(e_1) = R^q(e_1).$$

In view of (2),

$$U[L_k] = L_{(k+q) \pmod n}, \quad k = 0, \dots, n-1,$$

and consequently

$$U^i[L_k] = L_{(k+iq) \pmod n}, \quad i \in \mathbb{Z}, \quad k = 0, \dots, n-1. \quad (7)$$

In particular

$$U^n[L_k] = L_k, \quad k = 0, \dots, n-1.$$

Similarly

$$R^q[L_k] = L_{(k+q) \pmod n}, \quad k = 0, \dots, n-1.$$

Hence

$$H[L_k] = L_{(k+q) \pmod n}, \quad k = 0, \dots, n-1,$$

and consequently

$$H^i[L_k] = L_{(k+iq) \pmod n}, \quad i \in \mathbb{Z}, \quad k = 0, \dots, n-1. \quad (8)$$

Since  $L_{iq \pmod n} \subset S^1 \setminus L_0$  for  $i = 1, \dots, n-1$  we get by (4),

$$H^i(z) = R^{iq} \circ U^n(z) \quad \text{for } z \in L_0 \text{ and } i = 1, \dots, n-1.$$

Putting  $i = n$  we have

$$H^n(z) = R^{nq} \circ U^n(z) = U^n(z), \quad z \in L_0. \quad (9)$$

Define

$$\Psi(z) := U^{k-n} \circ H^{n-k}(z), \quad z \in L_{kq \pmod n}, \quad k = 0, \dots, n-1. \quad (10)$$

Note that (7) and (8) imply

$$\Psi[L_{kq \pmod n}] = L_{kq \pmod n}, \quad k = 0, \dots, n-1.$$

Hence  $\Psi$  is an orientation-preserving homeomorphism. Let  $z \in L_{kq \pmod n}$  for  $0 \leq k < n$ . Then  $H(z) \in L_{(k+1)q \pmod n}$ . Thus we have

$$\Psi(H(z)) = U^{k+1-n} \circ H^{n-k-1} \circ H(z) = U \circ U^{k-n} \circ H^{n-k}(z) = U(\Psi(z)).$$

Since  $\bigcup_{k=0}^{n-1} L_{kq \pmod n} = S^1$  we get (5). Moreover, by (9) and (10) we have

$$\Psi(z) = U^{-n} \circ H^n(z) = z \quad \text{for } z \in L_0. \quad \square$$

In the further considerations the crucial role will be played by the Babbage homeomorphisms.

Now we prove the following main result.

**Theorem 5.** *Let  $F : S^1 \rightarrow S^1$  be an orientation-preserving homeomorphism possessing periodic points of order  $n$ . Then for every  $a \in \text{Per } F$  there exists a unique pair of homeomorphisms  $T, G : S^1 \rightarrow S^1$  such that  $T^n = \text{id}$  and  $G(z) = z$  for  $z \in \overrightarrow{[F^p(a), a]}$ , where  $p = \text{char } F$ , satisfying the relation*

$$F = T^q \circ G \quad (11)$$

with  $q = n\alpha(F)$ .

**Proof.** Let  $a \in \text{Per } F$  and let  $U$  and  $\Phi$  be the homeomorphisms defined in Lemma 3. Define also an arbitrary function

$$V(z) := \begin{cases} U^n(z), & z \in L_0, \\ z, & z \in S^1 \setminus L_0. \end{cases}$$

We have

$$H = R^q \circ V. \quad (12)$$

By Lemma 4 there exists a homeomorphism  $\Psi : S^1 \rightarrow S^1$  such that

$$\Psi \circ H = U \circ \Psi.$$

Hence by (12) and (3),

$$\Psi \circ R^q \circ V = \Phi \circ F \circ \Phi^{-1} \circ \Psi,$$

so

$$R^q \circ V = (\Psi^{-1} \circ \Phi) \circ F \circ (\Psi^{-1} \circ \Phi)^{-1}.$$

Putting  $\Gamma := \Psi^{-1} \circ \Phi$  we have

$$F = \Gamma^{-1} \circ R^q \circ V \circ \Gamma = (\Gamma^{-1} \circ R^q \circ \Gamma) \circ (\Gamma^{-1} \circ V \circ \Gamma).$$

Thus

$$F = T^q \circ G,$$

where  $T := \Gamma^{-1} \circ R^q \circ \Gamma$  and  $G := \Gamma^{-1} \circ V \circ \Gamma$ . It is obvious that  $T^n = \text{id}$  and  $\alpha(T) = \frac{1}{n}$ .

By (1),  $\Phi(a) = e_0$ ,  $\Phi(F^p(a)) = e_1$  and by (6),  $\Psi(e_0) = e_0$  and  $\Psi(e_1) = e_1$ . Hence

$$\Gamma(a) = \Psi^{-1}(\Phi(a)) = \Psi^{-1}(e_0) = e_0$$

and

$$\Gamma(F^p(a)) = \Psi^{-1}(\Phi(F^p(a))) = \Psi^{-1}(e_1) = e_1.$$

If  $z \in \overrightarrow{[F^p(a), a]}$ , then  $\Gamma(z) \in \overrightarrow{[\Gamma(F^p(a)), \Gamma(a)]} = \overrightarrow{[e_1, e_0]}$ . Hence

$$G(z) = \Gamma^{-1} \circ V \circ \Gamma(z) = \Gamma^{-1}(\Gamma(z)) = z,$$

since  $V(u) = u$  for  $u \in \overrightarrow{[e_1, e_0]}$ . Thus  $G(z) = z$  for  $z \in \overrightarrow{[F^p(a), a]}$ .

Putting  $G_0 := G|_{I_0}$ , where  $I_0 = \overrightarrow{[a, F^p(a)]}$  we can write

$$G(z) := \begin{cases} G_0(z), & z \in I_0, \\ z, & z \in S^1 \setminus I_0. \end{cases}$$

To prove the uniqueness suppose

$$F = S^q \circ \widehat{G},$$

where  $S^n = \text{id}$ ,

$$\widehat{G}(z) := \begin{cases} G_1(z), & z \in I_0, \\ z, & z \in S^1 \setminus I_0, \end{cases}$$

and  $G_1 : I_0 \rightarrow I_0$  is an orientation-preserving homeomorphism. Note that

$$F^i[I_0] = I_{iq \pmod n}, \quad i = 0, 1, \dots, n, \quad (13)$$

where  $I_k = \overrightarrow{[F^{kp}(a), F^{(k+1)p}(a)]}$ . In fact  $F(F^{pk}(a)) = F^{p(k+q)}(a)$ , since  $pq = 1 \pmod n$  and  $F^n(a) = a$ . Hence

$$F[I_k] = I_{(k+q) \pmod n}, \quad k = 0, \dots, n-1. \quad (14)$$

Inductively we obtain (13). Since  $iq \not\equiv 0 \pmod n$  for  $i = 1, \dots, n-1$ , it follows from (13), that  $F^i(a) \notin I_0$  for  $i = 1, \dots, n-1$ . We shall show that

$$F^k(a) = S^{kq}(a), \quad k \in \mathbb{N}. \quad (15)$$

For  $k = 1$ ,  $F^1(a) = F(a) = S^q(G_1(a)) = S^q(a)$ . Suppose that (15) holds for  $k < n-1$ . Then we have

$$F^{k+1}(a) = F(F^k(a)) = S^q \circ \widehat{G}(F^k(a)) = S^q(F^k(a)) = S^q(S^{kq}(a)) = S^{(k+1)q}(a),$$

since  $F^k(a) \in I_0$  and  $\widehat{G}(F^k(a)) = F^k(a)$ . For  $k = n$  (15) is obvious. Since  $F^n(a)$  and  $S^n(a) = a$  we obtain (15) also for  $k > n$ . By putting  $kp$  instead of  $k$  in (15), we get

$$F^{kp}(a) = S^{kpq}(a) = S^k(a),$$

since  $pq = 1 \pmod n$ . Hence  $I_k = \overrightarrow{[S^k(a), S^{k+1}(a)]} = S^k[\overrightarrow{[a, S(a)]]}$ . Since  $F(a) = S^q(a)$  we get

$$I_k = S^k[I_0], \quad k = 0, \dots, n-1. \quad (16)$$

Now we shall show inductively that

$$F^k(z) = S^{kq}(G_1(z)) \quad \text{for } z \in I_0 \text{ and } k = 0, \dots, n-1. \quad (17)$$

For  $k = 1$  this is obvious. Suppose that (17) holds for  $k < n-1$  and let  $z \in I_0$ . By (16),

$$S^{kq}(G_1(z)) \in I_{qk} \subset S^1 \setminus I_0,$$

since  $G_1(z) \in I_0$ . Hence by (17),

$$F^{k+1}(z) = F(F^k(z)) = S^q \circ \widehat{G}(S^{kq}(G_1(z))) = S^q(S^{kq}(G_1(z))) = S^{(k+1)q}(G_1(z)),$$

since  $\widehat{G}(z) = z$  for  $z \in S^1 \setminus \text{Int } I_0$ . By putting  $k = n$  in (17), we get

$$F^n(z) = S^{nq}(G_1(z)) = G_1(z) \quad \text{for } z \in I_0, \quad (18)$$

because  $S^n = \text{id}$ . Similarly we prove that  $F^n(z) = G_0(z)$  for  $z \in I_0$ . Hence  $\widehat{G} = G$  and

$$F = S^q \circ G.$$

We may write  $S^q = F \circ G$  and  $S = S^{pq} = (F \circ G^{-1})^p$ . Similarly  $T = (F \circ G^{-1})^p$ . Thus  $T = S$ .  $\square$

From the proof of uniqueness we obtain also the following results.

**Corollary 6.** *If  $\alpha(F) = \frac{q}{n}$ ,  $\gcd(q, n) = 1$ , then for every  $a \in \text{Per } F$  there exists a unique Babbage homeomorphism  $T$  of order  $n$  such that*

$$F(z) := \begin{cases} T^q(F^n(z)), & z \in \overrightarrow{[a, F^p(a)]}, \\ T^q(z), & z \in S^1 \setminus \overrightarrow{[a, F^p(a)]}, \end{cases}$$

where  $p = \text{char } F$ .

**Proof.** This is a consequence of (18).  $\square$

**Remark 7.** Let  $\alpha(F) = \frac{q}{n}$ ,  $\gcd(q, n) = 1$ ,  $a \in \text{Per } F$ ,  $I_k = \overrightarrow{[F^{kp}(a), F^{(k+1)p}(a)]}$  for  $k = 0, \dots, n-1$ , and  $F = T^q \circ G$ , where  $p = \text{char } F$ ,  $T^n = \text{id}$  and  $G(z) = z$  for  $z \in S^1 \setminus I_0$ . Then

$$T[I_k] = I_{k+1 \pmod{n}}, \quad k = 0, \dots, n-1.$$

**Proof.** The assertion is a simple consequence of (16) and the equality  $T = S$ .  $\square$

**3.** In this section we deal with the iterates of the circle homeomorphisms. We shall show that the factorization theorem implies that the iterates of a homeomorphism possessing periodic points depend only on an interval homeomorphism and the iterates of its Babbage function. To describe the above property we use the following simple result from the number theory.

**Lemma 8.** *If  $1 \leq q < n$ ,  $\gcd(q, n) = 1$  and  $m \in \{0, \dots, n-1\}$ , then there exists a unique  $i_m \in \{0, \dots, n-1\}$  such that*

$$(m + i_m q) \pmod{n} = 0. \quad (19)$$

Moreover,  $i_m = -mp \pmod{n} = m(n-p) \pmod{n}$ , where  $pq = 1 \pmod{n}$ .

The proof of the above lemma is elementary.

**Theorem 9.** *Let  $\alpha(F) = \frac{q}{n}$ ,  $\gcd(q, n) = 1$ ,  $a \in \text{Per } F$  and*

$$I_m = \overrightarrow{[F^{mp}(a), F^{(m+1)p}(a)]}, \quad m = 0, 1, \dots, n-1. \quad (20)$$

*Then there exist a homeomorphism  $G: I_0 \rightarrow I_0$  and a Babbage function  $T$  of order  $N$  such that*

$$F_{|I_m}^{k+jn} = T^{qk} \circ \begin{cases} T^m \circ G_0^{j+1} \circ T_{|I_m}^{-m} & \text{if } i_m \leq k-1, \\ T^m \circ G_0^j \circ T_{|I_m}^{-m} & \text{if } i_m > k-1, \end{cases} \quad (21)$$

for  $m = 0, 1, \dots, n-1$ ,  $k = 0, 1, \dots, n$  and  $j \in \mathbb{N}$ .

In particular,

$$F_{|I_m}^k = T^{qk} \circ \begin{cases} T^m \circ G_0 \circ T_{|I_m}^{-m} & \text{if } i_m \leq k-1, \\ \text{id}_{|I_m} & \text{if } i_m > k-1, \end{cases} \quad (22)$$

for  $m = 0, 1, \dots, n-1$ ,  $k = 0, 1, \dots, n$ ,

$$F_{|I_m}^n = T^m \circ G_0 \circ T_{|I_m}^{-m}, \quad m = 0, 1, \dots, n-1, \quad (23)$$

and

$$F_{|I_0}^n = G_0. \quad (24)$$

**Proof.** Notice that

$$F^k[I_m] = I_{(m+kq) \pmod n}, \quad k = 0, 1, \dots, n, \quad m = 0, 1, \dots, n-1. \quad (25)$$

This formula is a simple consequence of (14). First we shall prove (22). The case  $k = 1$  follows directly from Theorem 5. In fact,  $i_m = 0$  if and only if  $m = 0$ . Thus we have

$$F|_{I_m} := T^q \circ \begin{cases} G_0, & m = 0, \\ \text{id}|_{I_m}, & m \neq 0. \end{cases} \quad (26)$$

Suppose that (22) holds for  $1 \leq k < n$ . We consider three cases.

Let  $i_m = k$ . Then by (25) and (19),  $F^k[I_m] = I_0$ . Hence by (26) and (22) we have  $F|_{I_m} = T^q_{|I_m}$  and  $F^k|_{I_m} = T^{kq}_{|I_m}$  since  $i_m > k - 1$ . Moreover,

$$\begin{aligned} F^{k+1}|_{I_m} &= F|_{I_0} \circ F^k|_{I_m} = T^q \circ G_0 \circ F^k|_{I_m} = T^q \circ G_0 \circ T^{kq}_{|I_m} = T^q \circ T^{kq+m} \circ G_0 \circ T^{kq+m} \circ T^{-m}_{|I_m} \\ &= T^{(k+1)q} \circ T^m \circ G_0 \circ T^{-m}_{|I_m}, \end{aligned}$$

since  $T^{kq+m} = \text{id}$ . Thus (25) holds for  $k + 1$ , since  $i_m \leq (k + 1) - 1$ .

Now let  $i_m < k$ . Then  $m' := (m + kq) \pmod n \neq 0$  and by (25),  $F^k[I_m] \cap I_0 = \emptyset$ . Hence, in view of (25), (26) and (22) we have

$$F^{k+1}|_{I_m} = F|_{I_{m'}} \circ F^k|_{I_m} = T^q \circ F^k|_{I_m} = T^q \circ T^{kq} \circ T^m \circ G_0 \circ T^{-m}_{|I_m} = T^{(k+1)q} \circ T^m \circ G_0 \circ T^{-m}_{|I_m},$$

since  $i_m \leq (k - 1)$ . Thus (22) holds for  $k + 1$ , since  $i_m \leq (k + 1) - 1$ .

Finally let  $i_m > k$ . Then  $F^k[I_m] \cap I_0 = \emptyset$  and by (26) and (22) we get

$$F^{k+1}|_{I_m} = F|_{I_{m'}} \circ F^k|_{I_m} = T^q \circ F^k|_{I_m} = T^q \circ T^{kq} = T^{(k+1)q}_{|I_m},$$

since  $i_m > k - 1$ . This means that (22) holds for  $k + 1$  since  $i_m > (k + 1) - 1$ . Hence it follows that (22) holds for  $k = 0, 1, \dots, n$ .

Putting  $k = n$  in (22) we get (23), since  $i_m \leq n - 1$  for every  $m$ . Putting  $m = 0$  in (23) we obtain (24). Since  $F^n[I_m] = I_m$ , (23) implies that

$$F^{jn}|_{I_m} = T^m \circ G_0^j \circ T^{-m}_{|I_m} \quad \text{for } j \in \mathbb{N}.$$

Hence (21) is a simple consequence of (22).  $\square$

**4.** Now we give some application of the factorization theorem to the determination of iterative roots of homeomorphism possessing fixed points.

Let  $F$  be a periodic homeomorphism with period  $n \geq 2$ . Then  $F^n$  has only fixed points. We consider the following inverse problem. Let  $H : S^1 \rightarrow S^1$  be a homeomorphism with fixed points. When for a given  $n \geq 2$  there exists a homeomorphism  $F$  with periodic points of order  $n$  such that

$$H = F^n?$$

In other words, when  $H$  has iterative roots of order  $n$  which are homeomorphisms without fixed points? To solve this problem we apply Theorem 9. From this theorem we obtain the following

**Lemma 10.** If  $\alpha(F) = \frac{q}{n}$ ,  $\gcd(q, n) = 1$  and  $H = F^n$ , then there is a partition of  $S^1$  on  $n$  consecutive arcs  $J_0, \dots, J_{n-1}$  such that  $H[J_i] = J_i$ ,  $i = 0, \dots, n - 1$ , and the functions  $H_i := H|_{J_i}$  are mutually conjugate, i.e., there exist homeomorphisms  $T_i : J_i \rightarrow J_{i+1}$ ,  $i = 0, \dots, n - 2$ , such that

$$H_{i+1} = T_i \circ H_i \circ T_i^{-1}. \quad (27)$$

**Proof.** Let  $a \in \text{Per } F$  and put  $J_i = I_i$ , where  $I_i$  are defined by (20) for  $i = 0, \dots, n - 1$ . By (25) we have  $H[J_i] = J_i$  for  $i = 0, \dots, n - 1$ . Put  $T_i := T|_{J_i}$ , where  $T$  is the Babbage function determined in Theorem 5. By Remark 7,  $T_i : J_i \rightarrow J_{i+1}$  for  $i = 0, \dots, n - 1$ . Thus by (23) we have

$$H_{i+1} = H|_{J_{i+1}} = F^n|_{J_{i+1}} = T^{i+1} \circ G_0 \circ T^{-i-1}_{|J_{i+1}} = T|_{J_i} \circ T^i \circ G_0 \circ T^{-i}_{|J_i} \circ T^{-1}_{|J_{i+1}} = T_i \circ H_i \circ T_i. \quad \square$$

This simple condition that the restrictions  $H$  to  $J_i$  are mutually conjugate also suffice for the existence of periodic iterative roots. This property is described in the following

**Theorem 11.** Let  $H : S^1 \rightarrow S^1$  be an orientation-preserving homeomorphism with  $\text{Fix } F \neq \emptyset$ .  $H$  has iterative roots of  $n$ th order, where  $n \geq 2$ , possessing periodic points of period  $n$  if and only if there exist a partition of  $S^1$  on  $n$  consecutive arcs  $J_0, \dots, J_{n-1}$  such that  $H[J_i] = J_i$  for  $i = 0, \dots, n-1$  and orientation-preserving homeomorphisms  $T_i : J_i \rightarrow J_{i+1}$ ,  $i = 0, \dots, n-1$ , such that

$$H|_{J_{i+1}} = T_i \circ H|_{J_i} \circ T_i^{-1}, \quad i = 0, \dots, n-2. \quad (28)$$

For every such a partition and a system of homeomorphisms  $T_i$  and every integer  $1 \leq q < n$  with  $\gcd(q, n) = 1$  the function

$$F(z) := \begin{cases} T^q(H(z)), & z \in J_0, \\ T^q(z), & z \in S^1 \setminus J_0, \end{cases} \quad (29)$$

where

$$T(z) := \begin{cases} T_i(z), & z \in J_i, \quad i = 0, \dots, n-2, \\ T_0^{-1} \circ T_1^{-1} \circ \dots \circ T_{n-1}^{-1}, & z \in J_{n-1}, \end{cases} \quad (30)$$

is an iterative root of  $H$  of  $n$ th order such that  $\text{Per } F \neq \emptyset$  and  $\alpha(F) = \frac{q}{n}$ . Formulas (29) and (30) determine the general form of all periodic iterative roots of order  $n$  possessing periodic points of order  $n$ .

**Proof.** The necessity of the condition follows directly from Lemma 10. To prove its sufficiency let us note that

$$T[J_i] = J_{i+1}, \quad i = 0, \dots, n-2, \quad (31)$$

and  $T^n = \text{id}$  (see e.g. [5]). It follows from (28) that

$$H|_{J_m} = T_{m-1} \circ \dots \circ T_0 \circ H|_{J_0} \circ T_0^{-1} \circ \dots \circ T_{m-1}^{-1} = T^m \circ H|_{J_0} \circ T|_{J_m}^{-m}. \quad (32)$$

Let  $z_0, \dots, z_{n-1} \in \text{Per } H$  be such that  $J_k = \overrightarrow{[z_k, z_{k+1}]}$ ,  $k = 0, \dots, n-1$ . Since  $\{J_k\}$  are consecutive arcs we have

$$\text{Arg} \frac{z_i}{z_0} < \text{Arg} \frac{z_{i+1}}{z_0} < 2\pi, \quad i = 0, \dots, n-2.$$

By (29) and (31) we have  $F(z_0) = T^q(z_0) = z_q$ , since  $H(z_0) = z_0$ . Further by (29) we have  $F^k(z_0) = T^{kq}(z_0) = z_{kq \pmod n}$ ,  $k = 1, \dots, n$ . Hence  $z_0 \in \text{Per}(F, n)$  and by Lemma 1,  $\alpha(F) = \frac{q}{n}$ . By (22) we may write  $F$  in the form  $F = T^q \circ G$ , where

$$G(z) := \begin{cases} H(z), & z \in J_0, \\ z, & z \in S^1 \setminus J_0. \end{cases}$$

By the uniqueness of the decomposition of  $F$  and by formula (23) in Theorem 9 we obtain

$$F|_{J_m}^n = T^m \circ H|_{J_0} \circ T|_{J_m}^{-m}, \quad m = 0, \dots, n-1.$$

Hence by (32) we have  $F^n = H$ .

To prove the uniqueness suppose that  $\widehat{F}^n = H$ ,  $\alpha(\widehat{F}) = \frac{q}{n}$  and  $z_0 \in \text{Per } \widehat{F}$ . Put  $J_i = \overrightarrow{[\widehat{F}^{ip}(z_0), \widehat{F}^{(i+1)p}(z_0)]}$  for  $i = 0, 1, \dots, n-1$ , where  $pq \equiv 1 \pmod n$ . Theorem 5 implies that

$$\widehat{F}(z) := \begin{cases} \widehat{T}^q(G_0(z)), & z \in J_0, \\ \widehat{T}^q(z), & z \in S^1 \setminus J_0, \end{cases}$$

where  $\widehat{T}^n = \text{id}$  and  $\alpha(\widehat{T}) = \frac{1}{n}$ . By Theorem 9 (formula (24)),  $G_0 = \widehat{F}|_{J_0}^n = H|_{J_0}$ . Thus we get (29) with  $T = \widehat{T}$ . Moreover,  $\widehat{T}$  is given by (30) (see [5]).  $\square$



## References

- [1] M. Bajger, On the structure of some flows on the unit circle, *Aequationes Math.* 55 (1998) 106–121.
- [2] K. Ciepliński, On the embedability of a homeomorphism of the circle in disjoint iteration group, *Publ. Math. Debrecen* 55 (1999) 363–383.
- [3] K. Ciepliński, M.C. Zdun, On a system of Schröder equations on the circle, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 13 (2003) 1883–1888.
- [4] I.P. Cornfeld, S.V. Fomin, Ya.G. Sinai, *Ergodic Theory*, Springer-Verlag, New York, 1982.
- [5] W. Jarczyk, Babagge equation on the circle, *Publ. Math. Debrecen* 63 (3) (2003) 389–400.
- [6] W. de Melo, S. van Strien, *One-Dimensional Dynamics*, *Ergeb. Math. Grenzgeb.* (3), Band 25, Springer-Verlag, Berlin, 1993.
- [7] P. Solarz, On some properties of orientation-preserving surjections on the circle, *Math. Slovaca* 57 (6) (2007) 1–14.
- [8] M.C. Zdun, On conjugacy of homeomorphisms of the circle possessing periodic points, *J. Math. Anal. Appl.* 330 (2007) 51–65.